

LOCAL WELL-POSEDNESS OF THE KDV EQUATION WITH QUASI PERIODIC INITIAL DATA

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ABSTRACT. We prove the local well-posedness for the Cauchy problem of the Korteweg-de Vries equation in a quasi periodic function space. The function space contains functions such that $f = f_1 + f_2 + \dots + f_N$ where f_j is in the Sobolev space of order $s > -1/2N$ of $2\pi\alpha_j^{-1}$ periodic functions. Note that f is not a periodic function when the ratio of periods α_i/α_j is irrational. The main tool of the proof is the Fourier restriction norm method introduced by Bourgain. We also prove an ill-posedness result in the sense that the flow map (if it exists) is not C^2 , which is related to the Diophantine problem.

1. INTRODUCTION

We consider the Cauchy problem of the Korteweg-de Vries equation as follows:

$$u_t + u_{xxx} + (u^2)_x = 0, \quad u(0) = f. \quad (1.1)$$

where $u(x, t)$ and $f(x)$ are complex or real valued functions. The local and global well-posedness for (1.1) in $H^s(\mathbb{T})$ has been intensively studied. In the present paper, we show the local well-posedness in a quasi periodic function space. Note that a 2π -periodic function on \mathbb{R} can be written as $f = \sum_{j=1}^{\infty} a_j e^{i\lambda_j x}$ with $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{Z}$. On the other hand, when f is an almost periodic function, the distribution of $\{\lambda_j\}_{j=1}^{\infty}$ on \mathbb{R} can be totally arbitrary. When f is a quasi periodic function, there exist $N \in \mathbb{N}$, $\alpha \in \mathbb{R}^N$ such that $\{\lambda_j\}_{j=1}^{\infty} \subset \{\alpha \cdot k : k \in \mathbb{Z}^N\}$. First, we recall some known results of the well-posedness for (1.1) in $H^s(\mathbb{T})$. In [3], Bourgain proved the local and global well-posedness with $s \geq 0$ by introducing the Fourier restriction norm method. In [18], Kenig, Ponce and Vega proved refined bilinear estimates to extend Bourgain's local well-posedness result to $s \geq -1/2$. In [6], Colliander, Keel, Staffilani, Takaoka and Tao proved the global well-posedness with $s \geq -1/2$. Since the KdV equation on H^s with $s < 0$ has no conservation law, it seemed difficult to consider the long time behavior of solutions in H^s with $s < 0$. To overcome this difficulty, they

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introduced a regularizing Fourier multiplier operator I and calculated a modified energy defined in H^s with $s \geq -1/2$, which is called the “ I -method”. Kappeler and Topalov proved the global well-posedness with $s \geq -1$ by the inverse scattering method in [15]. It is known that the flow map is analytic if and only if $s \geq -1/2$.

For the well-posedness results in $H^s(\mathbb{R})$, see [2], [17], [13], [3], [18], [6], [21] and [9]. For the ill-posedness results and counter examples of bilinear estimates related to this problem, see [18], [29], [25], [20], [5], [22] and [24]. To the best of author’s knowledge, there are few results for almost periodic initial data. In [8], Egorova proved that global solutions of (1.1) exist when the initial data f is real valued and $f \in Q_s(A)$ for $s \in \mathbb{N}$ and $s \geq 4$ and that the solutions are almost periodic in t and x . Here, we say $f \in Q_s(A)$ if and only if

$$\|f\|_s = \sup_x \left(\int_x^{x+1} |f|^2 + |\partial_x^s f|^2 dx \right)^{1/2} < \infty$$

and there exists a sequence of a_n -periodic functions $f_n \in H^s(\mathbb{R}/a_n)$ such that the following condition holds:

$$\forall p > 0, \quad \lim_{n \rightarrow \infty} e^{pa_{n+1}} \|f - f_n\|_s = 0, \quad (1.2)$$

where $A = \{a_n\}$ is an increasing sequence of positive numbers such that $a_{n+1}/a_n \in \mathbb{N} \setminus \{1\}$. Note that $f_1 + f_2 \notin Q_s(A)$ if f_j is an a_j -periodic function for $j = 1, 2$ and a_1/a_2 is irrational. Namely, quasi periodic functions are not in $Q_s(A)$. The Navier-Stokes equations with almost periodic initial data were studied in [10], [11], [30] and [12]. Kenig, Ponce and Vega studied the KdV with unbounded initial data in [19], which does not include almost periodic initial data.

In the proof of the local well-posedness results in $H^s(\mathbb{T})$ ([3], [18], [6]), the following argument is crucial. If u is a solution of (1.1) on \mathbb{T} , we have the following conservation law:

$$\int_{\mathbb{T}} u(t) dx = \int_{\mathbb{T}} f dx = c \quad (1.3)$$

for any t . Put

$$v = u - c. \quad (1.4)$$

Then, v satisfies

$$v_t + 2cv_x + v_{xxx} + (v^2)_x = 0, \quad v(0) = f - c \quad (1.5)$$

and $\int_{\mathbb{T}} v(t) dx = \int_{\mathbb{T}} v(0) dx = 0$. In the proof of the local well-posedness by the Fourier restriction norm method, the presence of the term $2cv_x$ in (1.5) does not

make any difference. Therefore, it is enough to study (1.1) under the assumption $\int_{\mathbb{T}} u(t) dx = \int_{\mathbb{T}} f dx = 0$ when $f \in H^s(\mathbb{T})$. When u is a spatially quasi periodic function, we have the following conservation law instead of (1.3):

$$\lim_{M \rightarrow \infty} \frac{1}{2M} \int_{-M}^M u(t) dx = \lim_{M \rightarrow \infty} \frac{1}{2M} \int_{-M}^M f dx = c$$

for any t and we can use the same argument as in the case of \mathbb{T} above. Therefore, we assume $\lim_{M \rightarrow \infty} \frac{1}{2M} \int_{-M}^M u(t) dx = \lim_{M \rightarrow \infty} \frac{1}{2M} \int_{-M}^M f dx = 0$ in the present paper.

We assume that $N \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{R}^N$ satisfy $\alpha \cdot k \neq 0$ for any $k \in \dot{\mathbb{Z}}^N$ where $\dot{\mathbb{Z}}^N = \mathbb{Z}^N \setminus (0, \dots, 0)$ throughout the paper. When f_j are $2\pi\alpha_j^{-1}$ periodic functions, $\sum_{j=1}^N f_j$ is not a periodic function but a quasi periodic function. It is a natural question to ask whether the local well-posedness holds or not in a function space which contains $\sum_{j=1}^N f_j$ with $f_j \in \dot{H}^{\sigma_j}(\mathbb{R}/2\pi\alpha_j^{-1}\mathbb{Z})$. A quasi periodic function $\sum_{j=1}^N f_j$ can be written as follows:

$$\sum_{j=1}^N \sum_{k \in \dot{\mathbb{Z}}} \widehat{f}_j(k) \exp(i\alpha_j k x)$$

where $\widehat{f}_j(k) : \dot{\mathbb{Z}} \rightarrow \mathbb{C}$ are the Fourier coefficients of f_j . Since the KdV equation has a quadratic term, we need to consider multiplications of functions of the form above. Therefore, we introduce the following function space.

Definition 1. For $\sigma = (\sigma_1, \dots, \sigma_N) \in \mathbb{R}^N$ and $a \in \mathbb{R}$, we define the Banach space $G^{\sigma, a}$ by

$$G^{\sigma, a} = \left\{ f(x) = \sum_{k \in \dot{\mathbb{Z}}^N} \widehat{f}(k) \exp(i\alpha \cdot kx) \mid \widehat{f} : \dot{\mathbb{Z}}^N \rightarrow \mathbb{C}, \|f\|_{G^{\sigma, a}} < \infty \right\} \quad (1.6)$$

where

$$\|f\|_{G^{\sigma, a}} := \|\widehat{f}\|_{\widehat{G}^{\sigma, a}} := \left\| |\alpha \cdot k|^a \prod_{j=1}^N \langle k_j \rangle^{\sigma_j} \widehat{f}(k) \right\|_{l^2(\dot{\mathbb{Z}}^N)}, \quad \langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}.$$

Note that the series in (1.6) does not converge even in \mathcal{S}' when σ_j are small (see Lemma 5.2). Therefore, we introduce the following notations and assumption.

Definition 2. Let e_j be the standard unit vector in \mathbb{R}^N whose j -th component is equal to 1 and the others are equal to 0. Put $d_j = \frac{s}{N-1} \{(1, 1, \dots, 1) - e_j\}$. For $s \in \mathbb{R}$, we define

$$\Lambda_s = \left\{ \sum_{j=1}^N \theta_j d_j \in \mathbb{R}^N \mid \sum_{j=1}^N \theta_j = 1, \theta_j \geq 0 \ (1 \leq j \leq N) \right\}. \quad (1.7)$$

Using the fact that Λ_s is the set of internal dividing points of $d_j (1 \leq j \leq N)$, we can easily check that the series in (1.6) converges in \mathcal{S}' when $a \leq 0$ and the following assumption holds:

$$(A) \quad s > (N - 1)/2, \quad \sigma \in \Lambda_s.$$

So, we assume (A) throughout the paper (except Lemma 5.2). Here, we give some remarks on (A), $G^{\sigma,a}$ and Λ_s .

Remark 1.1. If $f_j \in \dot{H}^{\sigma_j+a}(\mathbb{R}/2\pi\alpha_j^{-1}\mathbb{Z})$, then $\sum_{j=1}^N f_j \in G^{\sigma,a}$.

Remark 1.2. If $\sigma \in \Lambda_s$, then $\sum_{j=1}^N \sigma_j = s$.

Remark 1.3. For simpleness we assume $\sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_N$. Then, (A) is equivalent to that $\sigma_1 \geq 0$ and $\sigma_1 + \dots + \sigma_j > (j - 1)/2$ for any j satisfying $2 \leq j \leq N$.

We rewrite (1.1) into the following integral equation:

$$u(t) = e^{-t\partial_x^3} f + \int_0^t e^{-(t-t')\partial_x^3} \partial_x(u^2(t')) dt'. \quad (1.8)$$

Now, we state our main result.

Theorem 1.1. *Assume (A). Then, (1.8) is locally well-posed in $G^{\sigma,-1/2}$.*

Remark 1.4. If $\sigma_1 = \dots = \sigma_N$, then (A) is equivalent to $\sigma_j > 1/2 - 1/2N$. Therefore, from Remark 1.1, it follows that if $r > -1/2N$ and $f_j \in \dot{H}^r(\mathbb{R}/2\pi\alpha_j^{-1}\mathbb{Z})$ for $1 \leq j \leq N$, then $\sum_{j=1}^N f_j \in G^{\sigma,-1/2}$ with (A). Note that when $N = 1$, the lower bound $-1/2N$ coincides $-1/2$, which is the critical value to assure that the flow map in $H^s(\mathbb{T})$ is analytic. In this sense, Theorem 1.1 is a generalization of the local well-posedness result for the KdV on \mathbb{T} by Kenig, Ponce and Vega in [18].

For a more precise statement of this theorem, see Proposition 4.1. The main idea in the present paper is the definition of $G^{\sigma,a}$ and to take $a = -1/2$. It is known that a nonlinear interaction between high and very low frequency parts of solutions make the well-posedness problem difficult in the study of the KdV equation. To avoid this difficulty, we usually use the transform (1.4) for the periodic case. It is not enough for the quasi periodic case because the support of $\mathcal{F}_x[f](\xi) = \sum_{k \in \mathbb{Z}^N} \delta(\xi - \alpha \cdot k) \widehat{f}(k)$ can be dense around $\xi = 0$ when f is quasi periodic. In our proof, it is crucial to assume $\mathcal{F}_x[f](\xi)$ is very small around $\xi = 0$. This is the reason why we use the homogeneous weight function $|\alpha \cdot k|^a$ in the definition of $G^{\sigma,a}$ and take $a = -1/2$ (see Remark 5.1). Similar ideas were used for some different problems, for instance,

for a stochastic KdV equation in [7], for quadratic nonlinear Schrödinger equations in [23] and for the fifth order KdV equation in [16].

We also prove an ill-posedness result in the sense that the flow map in $G^{\sigma,a}$ (if it exists) is not C^2 at the origin under some condition on σ and a , which is related to the Diophantine problem (see Proposition 5.1). A relation between a Diophantine condition and the well-posedness of a coupled system of KdV equations is studied by Oh in [26] and [27]. The method used for the ill-posedness result is introduced by Bourgain in [4] to prove that the flow map of KdV in $H^s(\mathbb{T})$ (if it exists) with $s < -1/2$ can not be C^3 .

If u is a real valued solution of (1.1) and f is a quasi periodic function, the following conservation law holds (at least formally):

$$\|u(t)\|_{G^{0,0}}^2 = \lim_{M \rightarrow \infty} \frac{1}{2M} \int_{-M}^M u^2(t) dx = \lim_{M \rightarrow \infty} \frac{1}{2M} \int_{-M}^M f^2 dx = \|f\|_{G^{0,0}}^2.$$

Unfortunately, we can not extend the local solutions obtained in Theorem 1.1 to global ones by using this conservation law because $\|u(t)\|_{G^{\sigma,-1/2}}$ with the assumption (A) can not be bounded by $\|u(t)\|_{G^{0,0}}$. On the other hand, for the periodic case, we have the global well-posedness in $H^s(\mathbb{T})$ with $s \geq -1/2$ in [6]. Therefore, we can easily obtain the following result by combining the global well-posedness in $H^{\sigma_1-1/2}(\mathbb{R}/2\pi\alpha_1^{-1}\mathbb{Z})$ and Theorem 1.1.

Corollary 1.2. *Assume (A). Let $r > 0$ and $T > 0$. Then there exists $\varepsilon = \varepsilon(r, T) > 0$ which satisfies the following: if initial data $f \in G^{\sigma,-1/2}$ satisfies $f = g + h$, $g \in B_r(H^{\sigma_1-1/2}(\mathbb{R}/2\pi\alpha_1^{-1}\mathbb{Z}))$, $h \in B_\varepsilon(G^{\sigma,-1/2})$ and g is real valued, then the solution of (1.8) obtained in Theorem 1.1 can be extended on $t \in [-T, T]$.*

The main tool of the proof of Theorem 1.1 is the Fourier restriction norm method. We define the Fourier restriction norm as follows:

$$\begin{aligned} Z^{\sigma,a} &= \{u(t) \in \mathcal{S}'(\mathbb{R} : G^{\sigma,a}) \mid \|u\|_{Z^{\sigma,a}} < \infty\}, \\ \|u\|_{Z^{\sigma,a}} &= \|u\|_{X^{\sigma,a,1/2}} + \|u\|_{Y^{\sigma,a,0}}, \\ \|u\|_{X^{\sigma,a,b}} &= \|\langle \tau + (\alpha \cdot k)^3 \rangle^b \tilde{u}\|_{\widehat{G}^{\sigma,a} L_\tau^2}, \\ \|u\|_{Y^{\sigma,a,b}} &= \|\langle \tau + (\alpha \cdot k)^3 \rangle^b \tilde{u}\|_{\widehat{G}^{\sigma,a} L_\tau^1}, \end{aligned}$$

where $\tilde{u} = \mathcal{F}_t \hat{u}$. Note that $Z^{\sigma,a}$ is continuously embedded in $C_t(\mathbb{R} : G^{\sigma,a})$.

In the proof of Theorem 1.1, the following bilinear estimates play an important role.

Proposition 1.3. *Assume (A). Let $T > 0$. Then, there exists $\varepsilon > 1/8$ such that the following estimates hold for any u, v satisfying $\text{supp } u, \text{supp } v \subset [-T, T] \times \mathbb{R}$:*

$$\|(uv)_x\|_{X^{\sigma, -1/2, -1/2}} \lesssim T^\varepsilon \|u\|_{X^{\sigma, -1/2, 1/2}} \|v\|_{X^{\sigma, -1/2, 1/2}}, \quad (1.9)$$

$$\|(uv)_x\|_{Y^{\sigma, -1/2, -1}} \lesssim T^\varepsilon \|u\|_{X^{\sigma, -1/2, 1/2}} \|v\|_{X^{\sigma, -1/2, 1/2}}. \quad (1.10)$$

Finally, we give some notations. We will use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant C and write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. Implicit constants may depend on σ, b, α and N . For a Banach space X , We define $B_r(X) = \{f \in X \mid \|f\|_X \leq r\}$. Let $\chi(t)$ be a smooth function such that $\chi(t) = 1$ for $|t| < 1$ and $\chi(t) = 0$ for $|t| > 2$ and $\chi_T(t) = \chi(t/T)$. \mathcal{F}_t (resp. \mathcal{F}_x) is the Fourier transform with respect to t (resp. x). The rest of this paper is planned as follows. In Section 2, we give some notations and preliminary lemmas. In Section 3, we prove bilinear estimates. In Section 4, we prove Theorem 1.1 and Corollary 1.2. In Section 5, we prove an ill-posedness result.

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2. PRELIMINARY LEMMAS

Lemma 2.1. *Assume (A). Let $a \in \mathbb{R}$, $\varepsilon > \varepsilon' > 0$ and $T > 0$. Then, it follows that*

$$\|u\|_{X^{\sigma, a, 1/2-\varepsilon}} \lesssim T^{\varepsilon'} \|u\|_{X^{\sigma, a, 1/2}} \quad (2.1)$$

for any u satisfying $\text{supp } u \subset [-T, T] \times \mathbb{R}$.

Proof. Let $\text{supp } g(t) \subset [-T, T]$. Then, by the Plancherel theorem and the triangle inequality, we have

$$\|g\|_{H^{1/2-\varepsilon}} = \|\chi_T g\|_{H^{1/2-\varepsilon}} \lesssim \|(\langle \tau \rangle^{1/2-\varepsilon} |\mathcal{F}\chi_T|) * |\mathcal{F}g|\|_{L_\tau^2} + \| |\mathcal{F}\chi_T| * (\langle \tau \rangle^{1/2-\varepsilon} |\mathcal{F}g|) \|_{L_\tau^2}$$

By the Young inequality and the direct calculation, the first term is bounded by

$$\|\langle \tau \rangle^{1/2-\varepsilon} \mathcal{F}\chi_T\|_{L_\tau^p} \|\mathcal{F}g\|_{L_\tau^q} \lesssim T^{\varepsilon'} \|\langle \tau \rangle^{1/2} \mathcal{F}g\|_{L_\tau^2},$$

where $1/p = \varepsilon - \varepsilon' + 1/2$ and $1/q = 1 - \varepsilon + \varepsilon'$. In the same manner, the second term is bounded by

$$\|\mathcal{F}\chi_T\|_{L_\tau^p} \|\langle \tau \rangle^{1/2-\varepsilon} \mathcal{F}g\|_{L_\tau^q} \lesssim T^{\varepsilon'} \|\langle \tau \rangle^{1/2} \mathcal{F}g\|_{L_\tau^2},$$

where $1/p = 1 - \varepsilon'$ and $1/q = \varepsilon' + 1/2$. Therefore, we obtain

$$\begin{aligned} \|u\|_{X^{\sigma,a,1/2-\varepsilon}} &= \|\langle \partial_t \rangle^{1/2-\varepsilon} \exp(-t\partial_x^3)u\|_{G^{\sigma,a}L_t^2} \\ &\lesssim T^{\varepsilon'} \|\langle \partial_t \rangle^{1/2} \exp(-t\partial_x^3)u\|_{G^{\sigma,a}L_t^2} = T^{\varepsilon'} \|u\|_{X^{\sigma,a,1/2}}. \end{aligned}$$

□

Lemmas 2.2, 2.3 are variants of the Sobolev inequality.

Lemma 2.2. (i) Let $\min_{1 \leq j \leq N} \{\sigma_j\} > 1/2$. Then, it follows that

$$\|uv\|_{G^{\sigma,0}} \lesssim \|u\|_{G^{\sigma,0}} \|v\|_{G^{\sigma,0}}. \quad (2.2)$$

(ii) Let $\sigma_1 = 0$, $\min_{2 \leq j \leq N} \{\sigma_j\} > 1/2$ and $a > 1/4$. Then, it follows that

$$\left\| \sum_{k' \in \mathbb{Z}^N, k' \neq k} \langle |\alpha \cdot k| |\alpha \cdot (k - k')| |\alpha \cdot k'| \rangle^{-a} \widehat{u}(k - k') \widehat{v}(k') \right\|_{\widehat{G}^{\sigma,0}} \lesssim \|u\|_{G^{\sigma,0}} \|v\|_{G^{\sigma,0}}. \quad (2.3)$$

Proof. By the Schwarz inequality, we have

$$\begin{aligned} \|uv\|_{G^{\sigma,0}}^2 &\leq \sum_{k \in \mathbb{Z}^N} \prod_{j=1}^N \langle k_j \rangle^{2\sigma_j} \left| \sum_{k' \in \mathbb{Z}^N, k' \neq k} \widehat{u}(k - k') \widehat{v}(k') \right|^2 \\ &\leq C_1(\sigma) \sum_{k \in \mathbb{Z}^N} \sum_{k' \in \mathbb{Z}^N, k' \neq k} \prod_{j=1}^N \langle k_j - k'_j \rangle^{2\sigma_j} \langle k'_j \rangle^{2\sigma_j} \left| \widehat{u}(k - k') \widehat{v}(k') \right|^2 \\ &\leq C_1(\sigma) \|u\|_{G^{\sigma,0}}^2 \|v\|_{G^{\sigma,0}}^2, \end{aligned}$$

where

$$C_1(\sigma) = \sup_{k \in \mathbb{Z}^N} \sum_{k' \in \mathbb{Z}^N, k' \neq k} \prod_{j=1}^N \langle k_j \rangle^{2\sigma_j} \langle k_j - k'_j \rangle^{-2\sigma_j} \langle k'_j \rangle^{-2\sigma_j}.$$

Since $C_1(\sigma) < \infty$, we obtain (2.2). Put

$$F(k) = \sum_{k' \in \mathbb{Z}^N, k' \neq k} \langle |\alpha \cdot k| |\alpha \cdot (k - k')| |\alpha \cdot k'| \rangle^{-a} \widehat{u}(k - k') \widehat{v}(k'),$$

$$F_1(k) = F(k)|_{|\alpha \cdot k| \geq 1}, \quad F_2(k) = F(k)|_{|\alpha \cdot k| < 1}.$$

In the same manner as the proof of (2.2), we obtain

$$\|F_1\|_{\widehat{G}^{\sigma,0}} \lesssim \|u\|_{G^{\sigma,0}} \|v\|_{G^{\sigma,0}} \quad (2.4)$$

since

$$\begin{aligned}
& C_2(\sigma, a) \\
&= \sup_{|\alpha \cdot k| \geq 1} \sum_{k' \in \mathbb{Z}^N, k' \neq k} \langle |\alpha \cdot k| |\alpha \cdot (k - k')| |\alpha \cdot k'| \rangle^{-2a} \prod_{j=2}^N \langle k_j \rangle^{2\sigma_j} \langle k_j - k'_j \rangle^{-2\sigma_j} \langle k'_j \rangle^{-2\sigma_j} \\
&\lesssim \sum_{k' \in \mathbb{Z}^N, k' \neq k} \langle |\alpha \cdot (k - k')| |\alpha \cdot k'| \rangle^{-2a} \prod_{j=2}^N \min\{\langle k_j - k'_j \rangle, \langle k'_j \rangle\}^{-2\sigma_j} < \infty.
\end{aligned}$$

Because $\#\{k_1 \in \mathbb{Z} \mid |\alpha \cdot k| < 1\} \sim 1$ for any $(k_2, \dots, k_N) \in \mathbb{Z}^{N-1}$, we have

$$\begin{aligned}
\|F_2\|_{\dot{G}^{\sigma,0}}^2 &\lesssim \sum_{k \in \mathbb{Z}^N} \prod_{j=2}^N \langle k_j \rangle^{2\sigma_j} \left(\sum_{k' \in \mathbb{Z}^N, k' \neq k} |\widehat{u}(k - k')| |\widehat{v}(k')| \right)^2 \Big|_{|\alpha \cdot k| < 1} \\
&\lesssim \sup_{k_1 \in \mathbb{Z}} \sum_{(k_2, \dots, k_N) \in \mathbb{Z}^{N-1}} \prod_{j=2}^N \langle k_j \rangle^{2\sigma_j} \left(\sum_{k' \in \mathbb{Z}^N, k' \neq k} |\widehat{u}(k - k')| |\widehat{v}(k')| \right)^2.
\end{aligned} \tag{2.5}$$

By the Schwarz inequality, we have

$$\begin{aligned}
& \left(\sum_{k' \in \mathbb{Z}^N, k' \neq k} |\widehat{u}(k - k')| |\widehat{v}(k')| \right)^2 \\
&\lesssim \left(\sum_{(k'_2, \dots, k'_N) \in \mathbb{Z}^{N-1}} \|\widehat{u}(k - k')\|_{l_{k'_1}^2} \|\widehat{v}(k')\|_{l_{k'_1}^2} \right)^2 \\
&\lesssim \sum_{(k'_2, \dots, k'_N) \in \mathbb{Z}^{N-1}} \prod_{j=2}^N \langle k_j - k'_j \rangle^{-2\sigma_j} \langle k'_j \rangle^{-2\sigma_j} \\
&\quad \times \sum_{(k'_2, \dots, k'_N) \in \mathbb{Z}^{N-1}} \left(\prod_{j=2}^N \langle k_j - k'_j \rangle^{2\sigma_j} \langle k'_j \rangle^{2\sigma_j} \right) \|\widehat{u}(k - k')\|_{l_{k'_1}^2}^2 \|\widehat{v}(k')\|_{l_{k'_1}^2}^2.
\end{aligned} \tag{2.6}$$

Here we have

$$\sup_{(k_2, \dots, k_N) \in \mathbb{Z}^{N-1}} \sum_{(k'_2, \dots, k'_N) \in \mathbb{Z}^{N-1}} \prod_{j=2}^N \langle k_j \rangle^{2\sigma_j} \langle k_j - k'_j \rangle^{-2\sigma_j} \langle k'_j \rangle^{-2\sigma_j} \lesssim 1. \tag{2.7}$$

Therefore, from (2.5)–(2.7), we obtain

$$\begin{aligned}
& \|F_2\|_{\dot{G}^{\sigma,0}}^2 \\
&\lesssim \sum_{(k_2, \dots, k_N) \in \mathbb{Z}^{N-1}} \sum_{(k'_2, \dots, k'_N) \in \mathbb{Z}^{N-1}} \left(\prod_{j=2}^N \langle k_j - k'_j \rangle^{2\sigma_j} \langle k'_j \rangle^{2\sigma_j} \right) \|\widehat{u}(k - k')\|_{l_{k'_1}^2}^2 \|\widehat{v}(k')\|_{l_{k'_1}^2}^2 \\
&\lesssim \|u\|_{\dot{G}^{\sigma,0}}^2 \|v\|_{\dot{G}^{\sigma,0}}^2.
\end{aligned} \tag{2.8}$$

Collecting (2.4) and (2.8), we obtain (2.3). \square

Lemma 2.3. *Let $A, B, C \in \mathbb{R}$, $\min\{a, b, c\} \geq 0$, $\min\{a+b, b+c, c+a\} > 1/2$ and $a+b+c > 1$. Then, it follows that*

$$\|\langle \cdot + A \rangle^{-a} f * g\|_{L^1} \lesssim \|\langle \cdot + B \rangle^b f\|_{L^2} \|\langle \cdot + C \rangle^c g\|_{L^2} \quad (2.9)$$

where the implicit constant depends only on a, b and c .

Proof. (2.9) is equivalent to

$$\left\| \int \langle \tau + A \rangle^{-a} \langle \tau - \tau' + B \rangle^{-b} f(\tau - \tau') \langle \tau' + C \rangle^{-c} g(\tau') d\tau' \right\|_{L^1} \lesssim \|f\|_{L^2} \|g\|_{L^2}$$

By using the Schwarz inequality, the left hand-side is bounded by

$$\|\langle \tau + A \rangle^{-a} \langle \tau - \tau' + B \rangle^{-b} \langle \tau' + C \rangle^{-c}\|_{L^2_\tau L^2_{\tau'}} \|f(\tau - \tau') g(\tau')\|_{L^2_\tau L^2_{\tau'}} \lesssim \|f\|_{L^2} \|g\|_{L^2}.$$

Here, we used

$$\int \int \langle \tau + A \rangle^{-2a} \langle \tau - \tau' + B \rangle^{-2b} \langle \tau' + C \rangle^{-2c} d\tau' d\tau \leq C$$

where the constant depends only on a, b and c . \square

Lemma 2.4. *Let $\tau, \tau' \in \mathbb{R}$, $k, k' \in \mathbb{Z}^N$. Then, it follows that*

$$\begin{aligned} M &:= \max\{|\tau + (\alpha \cdot k)^3|, |\tau - \tau' + (\alpha \cdot (k - k'))^3|, |\tau' + (\alpha \cdot k')^3|\} \\ &\gtrsim |\alpha \cdot k| |\alpha \cdot (k - k')| |\alpha \cdot k'|. \end{aligned} \quad (2.10)$$

The proof of Lemma 2.4 follows from the triangle inequality and $a^3 - (a-b)^3 - b^3 = 3ab(a-b)$. The following lemmas are standard tools of the Fourier restriction norm. For the proofs, see e.g. [14].

Lemma 2.5. *Assume (A). Let $a \in \mathbb{R}$. Then, for any f , it follows that*

$$\|\chi(t) e^{-t\partial_x^3} f\|_{Z^{\sigma,a}} \lesssim \|f\|_{G^{\sigma,a}}.$$

Lemma 2.6. *Assume (A). Let $a \in \mathbb{R}$. Then, for any F , it follows that*

$$\|\chi(t) \int_0^t e^{-(t-t')\partial_x^3} F(t') dt'\|_{Z^{\sigma,a}} \lesssim \|F\|_{X^{\sigma,a,-1/2}} + \|F\|_{Y^{\sigma,a,-1}}.$$

3. PROOF OF THE BILINEAR ESTIMATES

Our aim in this section is to prove Proposition 1.3. The proof of the following lemma is almost similar to that of Theorem 1.2 in [18]. The difference is the presence of P, Q . It is crucial that the implicit constant in (3.1) does not depend on P, Q to apply Lemma 3.1 for the proof of Lemma 3.2.

Lemma 3.1. *Let $\alpha_1 \neq 0$ and $b > 3/8$. Then, the following estimate holds for any $P, Q \in \mathbb{R}$ and any functions $\tilde{u}(\tau, k_1), \tilde{v}(\tau, k_1)$ on $\mathbb{R} \times \mathbb{Z}$:*

$$\begin{aligned} & \| |\tilde{u}| *_{\tau, k_1} |\tilde{v}| \|_{\ell_{k_1}^2 L_\tau^2} \\ & \lesssim \| \langle \tau + (\alpha_1 k_1 + P)^3 \rangle^b \tilde{u}(\tau, k_1) \|_{\ell_{k_1}^2 L_\tau^2} \| \langle \tau + (\alpha_1 k_1 + Q)^3 \rangle^b \tilde{v}(\tau, k_1) \|_{\ell_{k_1}^2 L_\tau^2}, \end{aligned} \quad (3.1)$$

where the implicit constant depends only on α_1 and b .

Proof. We first consider the region $A = \{k_1 \in \mathbb{Z} \mid 3\alpha_1^2 |\alpha_1 k_1 + P + Q| > 1\}$. Applying the Schwartz inequality, we have

$$|\tilde{u}| *_{\tau, k_1} |\tilde{v}| \lesssim I(\tau, k_1)^{1/2} \left(\langle \tau + (\alpha_1 k_1 + P)^3 \rangle^{2b} |\tilde{u}|^2 *_{\tau, k_1} \langle \tau + (\alpha_1 k_1 + Q)^3 \rangle^{2b} |\tilde{v}|^2 \right)^{1/2}$$

where

$$I(\tau, k_1) = \langle \tau + (\alpha_1 k_1 + P)^3 \rangle^{-2b} *_{\tau, k_1} \langle \tau + (\alpha_1 k_1 + Q)^3 \rangle^{-2b}.$$

If we have

$$\sup_{\tau \in \mathbb{R}, k_1 \in A} I(\tau, k_1) \lesssim 1, \quad (3.2)$$

then we obtain

$$\begin{aligned} & \| |\tilde{u}| *_{\tau, k_1} |\tilde{v}| \|_{\ell_{k_1}^2(A) L_\tau^2} \\ & \lesssim \| (\langle \tau + (\alpha_1 k_1 + P)^3 \rangle^{2b} |\tilde{u}|^2) *_{\tau, k_1} (\langle \tau + (\alpha_1 k_1 + Q)^3 \rangle^{2b} |\tilde{v}|^2) \|_{\ell_{k_1}^1 L_\tau^1}^{1/2} \\ & \lesssim \| \langle \tau + (\alpha_1 k_1 + P)^3 \rangle^b \tilde{u} \|_{\ell_{k_1}^2 L_\tau^2} \| \langle \tau + (\alpha_1 k_1 + Q)^3 \rangle^b \tilde{v} \|_{\ell_{k_1}^2 L_\tau^2} \end{aligned}$$

from the Young inequality. Therefore, we have only to check (3.2). Since $\langle \tau + \beta \rangle^{-2b} *_{\tau} \langle \tau + \gamma \rangle^{-2b} \lesssim \langle \tau + \beta + \gamma \rangle^{\min\{1-4b, -2b\}}$, we have

$$I(\tau, k_1) \lesssim \sum_{k'_1 \in \mathbb{Z}} \langle \tau + (\alpha_1 k'_1 + Q)^3 + (\alpha_1(k_1 - k'_1) + P)^3 \rangle^{\min\{1-4b, -2b\}}.$$

Note that

$$\begin{aligned} & \tau + (\alpha_1 k'_1 + Q)^3 + (\alpha_1(k_1 - k'_1) + P)^3 \\ & = \tau + 3\alpha_1^2(\alpha_1 k_1 + P + Q)k'_1{}^2 + \Gamma(\alpha_1, k_1, k'_1, P, Q), \end{aligned}$$

and the order of $\Gamma(\alpha_1, k_1, k'_1, P, Q)$ with respect to k'_1 is less than 2. Because $\min\{1-4b, -2b\} < -1/2$, we obtain (3.2).

Next, we consider the region $A^c = \{k_1 \in \mathbb{Z} \mid 3\alpha_1^2 |\alpha_1 k_1 + P + Q| \leq 1\}$. Since $\#A^c \sim \alpha_1^{-3}$, it follows that

$$\begin{aligned} \|\tilde{u} *_{\tau, k_1} \tilde{v}\|_{\ell_{k_1}^2(A^c) L_\tau^2} &\lesssim \alpha_1^{-3} \sup_{k_1 \in \mathbb{Z}} \left\| \sum_{k'_1} |\tilde{u}(\tau, k_1 - k'_1)| *_{\tau} |\tilde{v}(\tau, k'_1)| \right\|_{L_\tau^2} \\ &\lesssim \alpha_1^{-3} \sup_{k_1 \in \mathbb{Z}} \sum_{k'_1} \left\| |\tilde{u}(\tau, k_1 - k'_1)| *_{\tau} |\tilde{v}(\tau, k'_1)| \right\|_{L_\tau^2}. \end{aligned} \quad (3.3)$$

Applying the Schwartz inequality, we have

$$\begin{aligned} &|\tilde{u}(\tau, k_1 - k'_1)| *_{\tau} |\tilde{v}(\tau, k'_1)| \\ &\lesssim J(\tau, k, k'_1)^{1/2} \left((\langle \tau + (\alpha_1(k_1 - k'_1) + P)^3 \rangle^{2b} |\tilde{u}(\tau, k_1 - k'_1)|^2) \right. \\ &\quad \left. *_{\tau} (\langle \tau + (\alpha_1 k'_1 + Q)^3 \rangle^{2b} |\tilde{v}(\tau, k'_1)|^2) \right)^{1/2} \end{aligned} \quad (3.4)$$

where

$$J(\tau, k_1, k'_1) = \langle \tau + (\alpha_1(k_1 - k'_1) + P)^3 \rangle^{-2b} *_{\tau} \langle \tau + (\alpha_1 k'_1 + Q)^3 \rangle^{-2b}.$$

Obviously,

$$\sup_{\tau \in \mathbb{R}, k_1 \in \mathbb{Z}, k'_1 \in A^c} J(\tau, k_1, k'_1) \lesssim 1. \quad (3.5)$$

Therefore, from (3.3)–(3.5), we obtain

$$\begin{aligned} &\|\tilde{u} *_{\tau, k_1} \tilde{v}\|_{\ell_{k_1}^2(A^c) L_\tau^2} \\ &\lesssim \alpha_1^{-3} \sup_{k_1 \in \mathbb{Z}} \sum_{k'_1} \left\| \langle \tau + (\alpha_1(k_1 - k'_1) + P)^3 \rangle^{2b} |\tilde{u}(\tau, k_1 - k'_1)|^2 \right. \\ &\quad \left. *_{\tau} \langle \tau + (\alpha_1 k'_1 + Q)^3 \rangle^{2b} |\tilde{v}(\tau, k'_1)|^2 \right\|_{L_\tau^1}^{1/2} \\ &\lesssim \alpha_1^{-3} \sup_{k_1 \in \mathbb{Z}} \sum_{k'_1} \left\| \langle \tau + (\alpha_1(k_1 - k'_1) + P)^3 \rangle^b |\tilde{u}(\tau, k_1 - k'_1)| \right\|_{L_\tau^2} \\ &\quad \times \left\| \langle \tau + (\alpha_1 k'_1 + Q)^3 \rangle^b |\tilde{v}(\tau, k'_1)| \right\|_{L_\tau^2} \\ &\lesssim \alpha_1^{-3} \|\langle \tau + (\alpha_1 k_1 + P)^3 \rangle^b \tilde{u}(\tau, k_1)\|_{\ell_{k_1}^2 L_\tau^2} \|\langle \tau + (\alpha_1 k_1 + Q)^3 \rangle^b \tilde{u}(\tau, k_1)\|_{\ell_{k_1}^2 L_\tau^2}. \end{aligned}$$

□

Lemma 3.2. *Assume (A) and $b > 3/8$. Then, the following bilinear estimates hold:*

$$\|uv\|_{X^{\sigma,0,0}} \lesssim \|u\|_{X^{\sigma,0,b}} \|v\|_{X^{\sigma,0,b}}, \quad (3.6)$$

$$\|uv\|_{X^{\sigma,0,-b}} \lesssim \|u\|_{X^{\sigma,0,b}} \|v\|_{X^{\sigma,0,0}}, \quad (3.7)$$

$$\|uv\|_{Y^{\sigma,0,-1/2}} \lesssim \|u\|_{X^{\sigma,0,b}} \|v\|_{X^{\sigma,0,b}}, \quad (3.8)$$

Proof. Since σ is an internal dividing point of $sd_j/(N-1)$, we have the estimates for general σ by interpolating among the estimates for $\sigma = sd_j/(N-1)$ with $j = 1, \dots, N$. By symmetry, we have only to consider the case $\sigma = sd_1/(N-1)$, namely, $\sigma_1 = 0, \sigma_2 = \dots = \sigma_N = s/(N-1) > 1/2$.

We first prove (3.6). The left hand-side of (3.6) is bounded by

$$\begin{aligned} & \left\| \prod_{j=2}^N \langle k_j \rangle^{\sigma_j} \sum_{k' \in \mathbb{Z}^N, k' \neq k} \int |\tilde{u}(\tau - \tau', k - k')| |\tilde{v}(\tau', k')| d\tau' \right\|_{\ell_k^2 L_\tau^2} \lesssim \\ & \left\| \prod_{j=2}^N \langle k_j \rangle^{\sigma_j} \sum_{k'_2 \dots k'_N} \left\| \sum_{k'_1} \int |\tilde{u}(\tau - \tau', k - k')| |\tilde{v}(\tau', k')| d\tau' \right\|_{\ell_{k_1}^2 L_\tau^2} \right\|_{\ell_{k_2 \dots k_N}^2}. \end{aligned}$$

Applying Lemma 3.1 with $P = \sum_{j=2}^N \alpha_j(k_j - k'_j)$, $Q = \sum_{j=2}^N \alpha_j k_j$ and (2.2) with respect to k_2, \dots, k_N variables, we obtain (3.6).

By the duality argument, (3.7) is equivalent to

$$\|uv\|_{X^{-\sigma,0,0}} \lesssim \|u\|_{X^{\sigma,0,b}} \|v\|_{X^{-\sigma,0,b}},$$

which is obtained in the same manner as the proof of (3.6) if we apply the duality of (2.2):

$$\|uv\|_{G^{-\sigma,0}} \lesssim \|u\|_{G^{\sigma,0}} \|v\|_{G^{-\sigma,0}}$$

instead of (2.2).

Finally, we prove (3.8). Divide $\mathbb{R} \times \mathbb{Z}^N \times \mathbb{R} \times \mathbb{Z}^N$ into the following three parts:

$$\begin{aligned} \Omega_1 &= \{(\tau, k, \tau', k') \mid |\tau + (\alpha \cdot k)^3| \geq \max\{|\tau' + (\alpha \cdot k')^3|, |\tau - \tau' + (\alpha \cdot (k - k'))^3|\}\} \\ \Omega_2 &= \{(\tau, k, \tau', k') \mid |\tau' + (\alpha \cdot k')^3| \geq \max\{|\tau + (\alpha \cdot k)^3|, |\tau - \tau' + (\alpha \cdot (k - k'))^3|\}\}, \\ \Omega_3 &= \{(\tau, k, \tau', k') \mid |\tau - \tau' + (\alpha \cdot (k - k'))^3| \geq \max\{|\tau' + (\alpha \cdot k')^3|, |\tau + (\alpha \cdot k)^3|\}\}. \end{aligned}$$

Put

$$\tilde{F}_j(u, v) = \sum_{k' \in \mathbb{Z}^N, k' \neq k} \int |\tilde{u}(\tau - \tau', k - k')| |\tilde{v}(\tau', k')| \mathbf{1}_{\Omega_j} d\tau' \quad (3.9)$$

for $j = 1, 2, 3$ where

$$\mathbf{1}_{\Omega}(\tau, k, \tau', k') = \begin{cases} 1 & \text{when } (\tau, k, \tau', k') \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$\|uv\|_{Y^{\sigma,0,-1/2}} \lesssim \sum_{j=1}^3 \|\tilde{F}_j(u, v)\|_{Y^{\sigma,0,-1/2}}.$$

By symmetry, we have only to estimate $\|F_j(u, v)\|_{Y^{\sigma,0,-1/2}}$ for $j = 1, 2$.

In Ω_1 , it follows that $|\tau + (\alpha \cdot k)^3| \gtrsim |\alpha \cdot k| |\alpha \cdot (k - k')| |\alpha \cdot k'|$ from Lemma 2.4. Therefore, we have

$$\begin{aligned} \|F_1(u, v)\|_{Y^{\sigma,0,-1/2}} &\lesssim \left\| \left\langle \tau + (\alpha \cdot k)^3 \right\rangle^{b-5/8} \sum_{k'} \int \langle |\alpha \cdot k| |\alpha \cdot (k - k')| |\alpha \cdot k'| \rangle^{1/8-b} \right. \\ &\quad \left. \times |\tilde{u}(\tau - \tau', k - k')| |\tilde{v}(\tau', k')| d\tau' \right\|_{\widehat{G}^{\sigma,0} L^1_\tau}, \end{aligned}$$

which is bounded by

$$\begin{aligned} &\left\| \sum_{k'} \langle |\alpha \cdot k| |\alpha \cdot (k - k')| |\alpha \cdot k'| \rangle^{1/8-b} \right. \\ &\quad \left. \times \|\langle \tau + (\alpha \cdot (k - k'))^3 \rangle^b \tilde{u}(\tau, k - k')\|_{L^2_\tau} \|\langle \tau + (\alpha \cdot k')^3 \rangle^b \tilde{v}(\tau, k')\|_{L^2_\tau} \right\|_{\widehat{G}^{\sigma,0}} \\ &\lesssim \|u\|_{X^{\sigma,0,b}} \|v\|_{X^{\sigma,0,b}} \end{aligned}$$

from Lemma 2.3 and Lemma 2.2.

In Ω_2 , it follows that $|\tau' + (\alpha \cdot k')^3| \geq |\tau + (\alpha \cdot k)^3|$. Take $\varepsilon > 0, b' > 3/8$ such that $b > b' + \varepsilon$. Then, we have

$$\begin{aligned} \|F_3(u, v)\|_{Y^{\sigma,0,-1/2}} &\leq \|\langle \tau + (\alpha \cdot k)^3 \rangle^{-1/2-\varepsilon} (|\tilde{u}| *_{\tau,k} \langle \tau + (\alpha \cdot k)^3 \rangle^\varepsilon |\tilde{v}|)\|_{\widehat{G}^{\sigma,0} L^1_\tau} \\ &\lesssim \| |\tilde{u}| *_{\tau,k} \langle \tau + (\alpha \cdot k)^3 \rangle^\varepsilon |\tilde{v}| \|_{\widehat{G}^{\sigma,0} L^2_\tau} \\ &\lesssim \|u\|_{X^{\sigma,0,b'}} \|v\|_{X^{\sigma,0,b'+\varepsilon}}. \end{aligned}$$

Here, we used the Schwartz inequality in the second line and (3.6) in the final line. \square

Proof of Proposition 1.3. First, we prove (1.9). By symmetry and the decomposition (3.9), we have only to prove

$$\|F_j(u, v)(\tau, k)\|_{X^{\sigma,1/2,-1/2}} \lesssim T^\varepsilon \|u\|_{X^{\sigma,-1/2,1/2}} \|v\|_{X^{\sigma,-1/2,1/2}} \quad (3.10)$$

for $j = 1, 2$. In Ω_1 , it follows that $\langle \tau + (\alpha \cdot k)^3 \rangle^{-1/2} |\alpha \cdot k|^{1/2} \lesssim |\alpha \cdot k'|^{-1/2} |\alpha \cdot (k - k')|^{-1/2}$. Therefore, for any $b > 3/8$, the left-hand side of (3.10) with $j = 1$ is bounded by

$$\|(|\alpha \cdot k|^{-1/2} |\tilde{u}|) * (|\alpha \cdot k|^{-1/2} |\tilde{v}|)\|_{X^{\sigma,0,0}} \lesssim \|u\|_{X^{\sigma,-1/2,b}} \|v\|_{X^{\sigma,-1/2,b}}$$

where we used (3.6). From Lemma 2.1, we obtain (3.10) with $j = 1$. In Ω_2 , it follows that $|\alpha \cdot k|^{1/2} \lesssim \langle \tau' + (\alpha \cdot k')^3 \rangle^{1/2} |\alpha \cdot k'|^{-1/2} |\alpha \cdot (k - k')|^{-1/2}$. Therefore, for any $b > 3/8$, the left-hand side of (3.10) with $j = 2$ is bounded by

$$\begin{aligned} &\|(|\alpha \cdot k|^{-1/2} |\tilde{u}|) *_{\tau,k} (|\alpha \cdot k|^{-1/2} |\tau' + (\alpha \cdot k')^3|^{1/2} |\tilde{v}|)\|_{X^{\sigma,0,-1/2}} \\ &\lesssim \|u\|_{X^{\sigma,-1/2,b}} \|v\|_{X^{\sigma,-1/2,1/2}} \end{aligned}$$

where we used (3.7). From Lemma 2.1, we obtain (3.10) with $j = 2$.

Next, we prove (1.10). By symmetry and the decomposition (3.9), we have only to prove

$$\|F_j(u, v)(\tau, k)\|_{Y^{\sigma, 1/2, -1}} \lesssim T^\varepsilon \|u\|_{X^{\sigma, -1/2, 1/2}} \|v\|_{X^{\sigma, -1/2, 1/2}} \quad (3.11)$$

for $j = 1, 2$. In Ω_1 , it follows that $\langle \tau + (\alpha \cdot k)^3 \rangle^{-1/2} |\alpha \cdot k|^{1/2} \lesssim |\alpha \cdot k'|^{-1/2} |\alpha \cdot (k - k')|^{-1/2}$. Therefore, for any $b > 3/8$, the left-hand side of (3.11) with $j = 1$ is bounded by

$$\|(|\alpha \cdot k|^{-1/2} |\tilde{u}|) *_{\tau, k} (|\alpha \cdot k|^{-1/2} |\tilde{v}|)\|_{Y^{\sigma, 0, -1/2}} \lesssim \|u\|_{X^{\sigma, -1/2, b}} \|v\|_{X^{\sigma, -1/2, b}}$$

where we used (3.8). From Lemma 2.1, we obtain (3.11) with $j = 1$. In Ω_2 , it follows that $|\alpha \cdot k|^{1/2} \lesssim \langle \tau' + (\alpha \cdot k')^3 \rangle^{1/2} |\alpha \cdot k'|^{-1/2} |\alpha \cdot (k - k')|^{-1/2}$. Therefore, for any $1/2 > b > 3/8$ the left-hand side of (3.11) with $j = 2$ is bounded by

$$\begin{aligned} & \|(|\alpha \cdot k|^{-1/2} |\tilde{u}|) *_{\tau, k} (|\alpha \cdot k|^{-1/2} |\tau + (\alpha \cdot k)^3|^{1/2} |\tilde{v}|)\|_{Y^{\sigma, 0, -1}} \\ & \lesssim \|(|\alpha \cdot k|^{-1/2} |\tilde{u}|) *_{\tau, k} (|\alpha \cdot k|^{-1/2} |\tau + (\alpha \cdot k)^3|^{1/2} |\tilde{v}|)\|_{X^{\sigma, 0, -b}} \\ & \lesssim \|u\|_{X^{\sigma, -1/2, b}} \|v\|_{X^{\sigma, -1/2, 1/2}} \end{aligned}$$

where we used the Schwarz inequality with respect to τ in the second line and (3.7) in the final line. From Lemma 2.1, we obtain (3.11) with $j = 2$. □

4. WELL-POSEDNESS RESULTS

In this section, we give outlines of the proofs of the local well-posedness result below and Corollary 1.2.

Proposition 4.1. *Assume (A).*

(Existence) Let $0 < \theta < 1/8$. For any $r > 0$, there exists $T \sim \min\{r^{-1/\theta}, 1\}$ which satisfies the following: for any $f \in B_r(G^{\sigma, -1/2})$, there exists $u \in C([-T, T]; G^{\sigma, -1/2}) \cap Z^{\sigma, -1/2}$ satisfying (1.8) on $t \in [-T, T]$. Moreover, the flow map, $f \in B_r(G^{\sigma, -1/2}) \mapsto u \in C([-T, T]; G^{\sigma, -1/2}) \cap Z^{\sigma, -1/2}$ is Lipschitz continuous.

(Uniqueness) If $u, v \in C([-T, T]; G^{\sigma, -1/2}) \cap Z^{\sigma, -1/2}$ satisfy (1.8) on $t \in [-T, T]$, then $u(t) = v(t)$ on $t \in [-T, T]$.

We can prove this proposition by the standard method of the Fourier restriction norm method (see e.g. [14], [18]). Therefore, we describe only the proof of the existence.

Proof. Put

$$M(u) = \chi(t)e^{-t\partial_x^3}f + \chi(t) \int_0^t e^{-(t-t')\partial_x^3} \partial_x (\chi_T(t')u(t'))^2 dt'.$$

We shall show M is a contraction on $B_{C_0}(Z^{\sigma, -1/2})$ where $C_0 > 0$ is to be determined later. Then, we have a solution of $u = M(u)$ by the fixed point argument, which satisfy (1.8) on $t \in [-T, T]$. From Lemma 2.5, we have

$$\|\chi(t)e^{-t\partial_x^3}f\|_{Z^{\sigma, -1/2}} \lesssim \|f\|_{G^{\sigma, -1/2}}.$$

From Lemma 2.6 and Proposition 1.3, we have

$$\|\chi(t) \int_0^t e^{-(t-t')\partial_x^3} \partial_x (\chi_T(t')u(t'))^2 dt'\|_{Z^{\sigma, -1/2}} \lesssim T^\theta \|\chi_T u\|_{Z^{\sigma, -1/2}}^2 \lesssim T^\theta \|u\|_{Z^{\sigma, -1/2}}^2.$$

Combining them, we obtain

$$\begin{aligned} \|Mu\|_{Z^{\sigma, -1/2}} &\leq C\|f\|_{G^{\sigma, -1/2}} + CT^\theta \|u\|_{Z^{\sigma, -1/2}}^2 \\ &\leq 2Cr \end{aligned}$$

if $u \in B_{C_0}(Z^{\sigma, -1/2})$, $C_0 = 2Cr$ and $T \leq (4C^2r)^{-1/\theta}$. We conclude that M is a map from $B_{C_0}(Z^{\sigma, -1/2})$ to itself. In the same manner, we have

$$\begin{aligned} \|M(u - v)\|_{Z^{\sigma, -1/2}} &\leq CT^\theta \|u - v\|_{Z^{\sigma, -1/2}} \|u + v\|_{Z^{\sigma, -1/2}} \\ &\leq 1/2 \|u - v\|_{Z^{\sigma, -1/2}} \end{aligned}$$

for $u, v \in B_{C_0}(Z^{\sigma, -1/2})$. □

Here, we introduce the localized norm of $Z^{\sigma, a}$;

$$\|u\|_{Z_{[t_1, t_2]}^{\sigma, a}} = \{\inf \|v\|_{Z^{\sigma, a}} \mid v = u \text{ on } [t_1, t_2]\}.$$

Outline of the proof of Corollary 1.2. In Theorem 2 in [6], a priori estimate and global well-posedness for (1.8) with real valued initial data in $\dot{H}^{\sigma_1 - 1/2}(\mathbb{R} \setminus 2\pi\alpha_j^{-1}\mathbb{Z})$ have been proved. Therefore, there exist $C_* = C_*(T, r)$, $\delta = \delta(T, r)$ and $1/8 > \theta > 0$ satisfying

$$\begin{aligned} \sup_{t \in [-T, T]} \|v(t)\|_{G^{\sigma, -1/2}} &= \sup_{t \in [-T, T]} \|v(t)\|_{H^{\sigma_1 - 1/2}} \leq C_*, \\ 0 < \delta^\theta &< C_*^{-1}, \quad \sup_{-k+1 \leq j \leq k} \|v\|_{Z_{[(j-1)\delta, j\delta]}^{\sigma, -1/2}} < C_*, \quad k := T/\delta \in \mathbb{N} \end{aligned}$$

for the solution v of (1.8) with initial data g . Since we have the local well-posedness in $G^{\sigma, -1/2}$ and the existence time of solutions depends only on the size of the norm

of initial data, we only need to prove the following a priori estimate: for sufficiently small $\varepsilon > 0$, the solution u of (1.8) satisfies

$$\sup_{t \in [-T, T]} \|u(t)\|_{G^{\sigma, -1/2}} \leq 2C_*.$$

For that purpose, put $w = u - v$ and we will show

$$\sup_{t \in [-T, T]} \|w(t)\|_{G^{\sigma, -1/2}} \leq C\varepsilon < C_* \quad (4.1)$$

where $\varepsilon > 0$ is to be determined later. For simpleness we consider only $0 \leq t \leq T$. w satisfies the following integral equation:

$$w(t) = e^{-t\partial_x^3} w((j-1)\delta) + \int_{(j-1)\delta}^t e^{-(t-t')\partial_x^3} \partial_x (w(t')(w(t') + 2v(t'))) dt'.$$

Therefore, if we have

$$\|w((j-1)\delta)\|_{G^{\sigma, -1/2}} \leq C_{j-1}\varepsilon < C_*,$$

then we obtain

$$\sup_{(j-1)\delta \leq t \leq j\delta} \|w(t)\| \leq C\|w\|_{Z_{[(j-1)\delta, j\delta]}^{\sigma, -1/2}} \leq C\|w((j-1)\delta)\|_{G^{\sigma, -1/2}} < C_j\varepsilon$$

in the same manner as the proof of Proposition 4.1 because $\delta^\theta \|w + 2v\|_{Z_{[(j-1)\delta, j\delta]}^{\sigma, -1/2}} \ll 1$. We can iterate this argument until $j = k$ if we take ε sufficiently small such that $\sup_{1 \leq j \leq k} C_j\varepsilon \ll C_*$. \square

5. ILL-POSEDNESS RESULTS

Let $\mu \in \mathbb{R}$, $\rho \in \mathbb{R}$ and $K > 0$. We say that μ is a number of type (K, ρ) if for any integers p and $q \neq 0$,

$$\left| \mu - \frac{p}{q} \right| \geq \frac{K}{|q|^{2+\rho}}.$$

Note that almost every $\mu \in \mathbb{R}$ is a number of type (K, ρ) for some $K > 0$ when $\rho > 0$ by Lemma 3 on p. 114 of [1]. By Dirichlet's Theorem (see, e.g. Theorem 1A on p. 34 of [28]), if $\mu \in \mathbb{R}$ is a number of type (K, ρ) for some $K > 0$, then $\rho \geq 0$. We define ρ_μ by

$$\rho_\mu = \inf \{ \rho \geq 0 \mid \mu \text{ is a number of type } (K, \rho) \text{ for some } K > 0 \}$$

as in [26].

For simplicity, we consider only the case $N = 2$.

Proposition 5.1. *Assume that $\mu = \alpha_1/\alpha_2$, $a > (2 \min\{\sigma_1, \sigma_2\} - \min\{1, \rho_\mu\})/(1 + \rho_\mu)$ and $\min\{\sigma_1, \sigma_2\} + a \geq -2$. Let $r > 0, 1 > T > 0$. Then, the flow map of (1.8) from initial data $f \in B_r(G^{\sigma,a})$ to a solution $u \in C([-T, T] : G^{\sigma,a})$ (if it exists) cannot be C^2 differentiable at the origin.*

Remark 5.1. When $\sigma_2 = 0, \rho_\mu = 1$, the assumption on a in Proposition 5.1 is $a > -1/2$. This is the reason why we take $a = -1/2$ in Theorem 1.1.

Proof. By symmetry, we assume $\sigma_1 \leq \sigma_2$. We assume that the flow map is C^2 and $u(\gamma, t, x)$ is the solution of (1.8) with initial data γf where $\gamma > 0$. Then, following the argument in [29] (see also [4]), we have

$$\begin{aligned} \frac{\partial u}{\partial \gamma}(0, t, x) &= e^{-t\partial_x^3} f(x) := u_1(t, x), \\ \frac{\partial^2 u}{\partial \gamma^2}(0, t, x) &= \int_0^t e^{-(t-t')\partial_x^3} \partial_x(u_1(t', x))^2 dt' := u_2(t, x). \end{aligned}$$

We only need to show that

$$\|u_2(t, x)\|_{G^{\sigma,a}} \lesssim \|f\|_{G^{\sigma,a}}^2$$

fails. From the definition of ρ_μ , for any $\varepsilon > 0$, we have the following:

(i) There exists $K > 0$ such that

$$\left| \frac{\alpha_1}{\alpha_2} - \frac{p}{q} \right| \geq \frac{K}{|q|^{2+\rho_\mu+\varepsilon}} \quad (5.1)$$

holds for any integers p and $q \neq 0$.

(ii) For any $n \in \mathbb{N}$, there exists $(p_n, q_n) \in \mathbb{Z}^2$ such that

$$\left| \frac{\alpha_1}{\alpha_2} - \frac{p_n}{q_n} \right| \leq \frac{n^{-1}}{|q_n|^{2+\rho_\mu-\varepsilon}}. \quad (5.2)$$

Note that, from (5.2), it follows that $p_n \sim q_n \rightarrow \infty$ as $n \rightarrow \infty$ if ε is sufficiently small. Put

$$\widehat{f}_n(k) = |q_n|^{-a-\sigma_1} \mathbf{1}_{(-q_n, 0)}(k) + |\alpha_1 q_n - \alpha_2 p_n|^{-a} |q_n|^{-\sigma_1} |p_n|^{-\sigma_2} \mathbf{1}_{(q_n, -p_n)}(k)$$

where $\mathbf{1}_{(a,b)}(k) = 1$ for $k = (a, b)$ and $\mathbf{1}_{(a,b)}(k) = 0$ otherwise.

Then,

$$\|f_n\|_{G^{\sigma,a}}^2 \sim 1.$$

We shall show that $\|u_2(t, x)\|_{G^{\sigma, a}} \rightarrow \infty$ as $n \rightarrow \infty$. Put $\varphi_\alpha(k, k') = 3(\alpha \cdot k)(\alpha \cdot k')(\alpha \cdot (k - k'))$. Then,

$$\begin{aligned} \widehat{e^{t\partial_x^3} u_2}(t, k) &= i \int_0^t \sum_{k' \in \mathbb{Z}^N} e^{-it' \varphi_\alpha(k, k')} (\alpha \cdot k) \widehat{f}_n(k - k') \widehat{f}_n(k') dt' \\ &= I_1 + I_2 + I_3 \end{aligned}$$

where

$$\begin{aligned} I_1 &= -2i \int_0^t e^{-it' \varphi_\alpha((0, -p_n), (q_n, -p_n))} dt' \\ &\quad \times (\alpha_2 p_n) |\alpha_1 q_n - \alpha_2 p_n|^{-a} |q_n|^{-a-2\sigma_1} |p_n|^{-\sigma_2} \mathbf{1}_{(0, -p_n)}(k), \\ I_2 &= -2i \int_0^t e^{-it' \varphi_\alpha((-2q_n, 0), (-q_n, 0))} dt' (\alpha_1 q_n) |q_n|^{-2a-2\sigma_1} \mathbf{1}_{(-2q_n, 0)}(k), \\ I_3 &= 2i \int_0^t e^{-it' \varphi_\alpha((2q_n, -2p_n), (q_n, -p_n))} dt' \\ &\quad \times (\alpha_1 q_n - \alpha_2 p_n) |\alpha_1 q_n - \alpha_2 p_n|^{-2a} |q_n|^{-2\sigma_1} |p_n|^{-2\sigma_2} \mathbf{1}_{(2q_n, -2p_n)}(k). \end{aligned}$$

We shall use the following inequality to estimate the integral with respect to t'

$$\frac{1}{\langle y \rangle} \gtrsim \sup_{t \in [-T, T]} \left| \int_0^t e^{-it' y} dt' \right| \gtrsim \frac{T}{\langle y \rangle}.$$

First, we estimate $\sup_{t \in [-T, T]} \|I_1\|_{G^{\sigma, a}}$. Since $|q_n|^{-\rho_\mu - 1 + \varepsilon} \gtrsim |\alpha_1 q_n - \alpha_2 p_n| \gtrsim |q_n|^{-\rho_\mu - 1 - \varepsilon}$, we have

$$\begin{aligned} \sup_{t \in [-T, T]} \|I_1\|_{G^{\sigma, a}} &\gtrsim T \frac{|\alpha_1 q_n - \alpha_2 p_n|^{-a} |q_n|^{-a-2\sigma_1} |p_n|^{a+1}}{\langle |\alpha_1 q_n - \alpha_2 p_n| |q_n| |p_n| \rangle} \\ &\gtrsim T \min\{|q_n|^{a(\rho_\mu + 1 \pm \varepsilon) - 2\sigma_1 + 1}, |q_n|^{(a+1)(\rho_\mu + 1 \pm \varepsilon) - 2\sigma_1 - 1}\}, \end{aligned}$$

which goes to ∞ for sufficiently small $\varepsilon > 0$.

Obviously, we have

$$\sup_{t \in [-T, T]} \|I_2\|_{G^{\sigma, a}} \lesssim \frac{|q_n|^{-a-\sigma_1+1}}{\langle |q_n|^3 \rangle} \sim |q_n|^{-a-\sigma_1-2} \lesssim 1.$$

Finally, we estimate $\sup_{t \in [-T, T]} \|I_3\|_{G^{\sigma, a}}$.

$$\begin{aligned} \sup_{t \in [-T, T]} \|I_3\|_{G^{\sigma, a}} &\lesssim \frac{|\alpha_1 q_n - \alpha_2 p_n|^{-a+1} |q_n|^{-\sigma_1} |p_n|^{-\sigma_2}}{\langle |\alpha_1 q_n - \alpha_2 p_n|^3 \rangle} \\ &<< \sup_{t \in [-T, T]} \|I_1\|_{G^{\sigma, a}}. \end{aligned}$$

Therefore, we conclude $\|u_2(t, x)\|_{G^{\sigma, a}} \rightarrow \infty$. \square

Finally we prove the following lemma to show that the condition $s > (N - 1)/2$ in (A) is necessary to assure that the series in (1.6) converges in \mathcal{S}' .

Lemma 5.2. *Let $a \in \mathbb{R}$, $\sigma_j \geq 0$ for all $0 \leq j \leq N$ and $\sum_{j=1}^N \sigma_j \leq (N-1)/2$. Then there exists $\{f_n\} \subset G^{\sigma,a}$ which satisfies the following:*

- (i) $\sup_n \|f_n\|_{G^{\sigma,a}} \lesssim 1$
- (ii) for some $\varphi \in \mathcal{D}$, $\langle f_n, \mathcal{F}_\xi^{-1}[\varphi] \rangle = \langle \mathcal{F}_x[f_n], \varphi \rangle \rightarrow \infty$ as $n \rightarrow \infty$

Proof. Put

$$A_n = \{x \in \mathbb{R}^N \mid \langle x \rangle \leq n \text{ and } 2|\alpha_N| \leq |\alpha \cdot x| \leq 4|\alpha_N|\},$$

$$\widehat{f}_n(k) = \begin{cases} \langle k \rangle^{1-N} (\log \langle k \rangle)^{-1}, & \text{when } k \in \mathbb{Z}^N \cap A_n \\ 0, & \text{otherwise.} \end{cases}$$

Let $\varphi(\xi) \in \mathcal{D}$ such that $\varphi(\xi) = 1$ for $2|\alpha_N| \leq \xi \leq 4|\alpha_N|$ and $\varphi(\xi) = 0$ for $\xi \leq |\alpha_N|$ or $\xi \geq 5|\alpha_N|$. We have the following relation between the Fourier transform and the Fourier coefficient

$$\mathcal{F}_x[f_n](\xi) = \sum_{k \in \mathbb{Z}^N} \delta(\xi - \alpha \cdot k) \widehat{f}_n(k).$$

Therefore, we have

$$\langle \mathcal{F}_x[f_n], \varphi \rangle \sim \sum_{k \in \mathbb{Z}^N \cap A_n} \langle k \rangle^{1-N} (\log \langle k \rangle)^{-1} \rightarrow \infty,$$

and

$$\begin{aligned} \|f_n\|_{G^{\sigma,a}}^2 &\sim |\alpha_N|^{2a} \sum_{k \in \mathbb{Z}^N \cap A_n} \langle k \rangle^{2(1-N)} (\log \langle k \rangle)^{-2} \prod_{j=1}^N \langle k_j \rangle^{2\sigma_j} \\ &\lesssim \sum_{k \in \mathbb{Z}^N \cap A_n} \langle k \rangle^{1-N} (\log \langle k \rangle)^{-2} \lesssim 1. \end{aligned}$$

Here we used

$$\lim_{n \rightarrow \infty} \int_{A_n} \frac{1}{\langle x \rangle^{N-1} \log \langle x \rangle} dx \sim C(\alpha_N) \int_{\mathbb{R}^{N-1}} \frac{1}{\langle x \rangle^{N-1} \log \langle x \rangle} dx = \infty$$

and

$$\sup_n \int_{A_n} \frac{1}{\langle x \rangle^{N-1} (\log \langle x \rangle)^2} dx \sim C(\alpha_N) \int_{\mathbb{R}^{N-1}} \frac{1}{\langle x \rangle^{N-1} (\log \langle x \rangle)^2} dx \lesssim 1.$$

□

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